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Squeezing of a coupled state of two spinors

A R Usha Devi[1], K S Mallesh[2], Mahmoud A A Sbaih[2], K B Nalini[3] and G Ramachandran[4]

[1] Department of Physics, Bangalore University, Bangalore 560 056, India
[2] Department of Studies in Physics, University of Mysore, Mysore 570 006, India
[3] Vidya Vikas Institute of Engineering and Technology, Mysore 570 010, India
[4] Indian Institute of Astrophysics, Koramangala, Bangalore 560 034, India

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Abstract

The notion of spin squeezing involves a reduction in the uncertainty of a component of the spin vector $\vec{S}$ below a certain limit. This aspect has been studied earlier (Mallesh *et al* 2000a *J. Phys. A: Math. Gen.* **33** 779, Mallesh *et al* 2000b *J. Phys. A: Math. Gen.* **34** 3293) for pure and mixed states of definite spin. In this paper, this study has been extended to coupled spin states which do not possess a sharp spin value. A general squeezing criterion has been obtained such that a direct product state for two spinors is not squeezed. The squeezing aspect of entangled states is studied in relation to their spin–spin correlations.

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1. Introduction

The notion of squeezing which involves reduction in the variance (uncertainty) of an observable below a standard quantum limit has been studied [3, 4] in the literature for an oscillator and a bosonic field. The squeezed states of the electromagnetic field have received due attention in the last decade and studies done [3, 4] so far on these have focused on various aspects such as the nonclassical features associated with them, on possible ways and means of generating them and on the practical applications of these states to achieve minimum noise in amplifiers and in optical interferometers. On the theoretical side, the conceptual basis which leads to these squeezed states of radiation field has also triggered the ideas of visualizing squeezing in non-canonical quantum systems and consequently the notion of squeezing has been extended to multilevel atomic states and ensembles [5–8] and to spin systems with arbitrary but sharp spin values [9, 10].

Quite recently, we have [1, 2] analysed in detail the notion of spin squeezing and looked into several aspects of squeezing in the case of oriented and non-oriented systems, and in the case of a coupled spin s system composed of $2s$ spinor states. Generalizing the squeezing criterion given by Kitagawa and Ueda [9], we have made a detailed study of pure as well as mixed states. Kitagawa and Ueda have suggested in their paper that the occurrence of squeezing is a consequence of quantum spin–spin correlations that exist among the $2s$ spinor states which together constitute the spin s state. Our study reveals that all oriented states are not squeezed but non-oriented states exhibit squeezing. In the case of a pure spin-1 state, our claim is that the notion of non-oriented is synonymous with the notion of squeezing.

In the light of the above studies on spin systems, it becomes relevant to extend the ideas of squeezing to bipartite systems which do not possess a sharp value of spin. Such systems can arise due to the coupling of two systems with sharp angular momenta. An additional aspect that arises in such a coupled state is whether a given state is entangled or not [11]. An entangled state cannot be written as a product of the spin states of the individual systems but only as a linear combination of such products. Further, it follows that the self and mutual spin–spin correlations will be present in an entangled state. It is therefore necessary to look for possible relationships among the three aspects, namely squeezing, entanglement and spin–spin correlations.

The present paper which addresses these intrinsic notions is organized as follows. In section 2, we look at the properties of coupled states and discuss the conditions to be satisfied by a coupled state to be a direct product state and an entangled state. In the next section, we take up the discussion on the squeezing aspect for a coupled state which may be either entangled or not. In the case of an uncoupled state, the mutual spin–spin correlation will be absent and consequently this has to be taken into account while defining the squeezing criterion. Taking this into consideration, we propose a generalization of the squeezing criterion for a coupled state of two spin systems. A detailed presentation of this is given in section 3. The dependence of squeezing on entanglement and on correlations is also explored here. Section 4 deals with the time evolution of a coupled pure spin state in the presence of a spin–spin interaction. Our aim here is to show that a coupled pure spin state undergoing such interaction develops squeezing as time elapses, even though, it may not have squeezing initially. We also look at the manner in which squeezing depends on spin–spin correlations. The last section is devoted to comments.

2. Properties of coupled states

The $s = \frac{1}{2}$ states enjoy an exalted status in quantum theory since an arbitrary spin $\frac{1}{2}$ state can always be looked upon either as a $\left|\frac{1}{2}\frac{1}{2}\right\rangle$ state or as a $\left|\frac{1}{2} - \frac{1}{2}\right\rangle$ state with respect to an appropriately chosen Cartesian frame. Thus, a spinor is always oriented with respect to some z-axis. Following the Schwinger construction [12], any higher spin $s > \frac{1}{2}$ can be realized in terms of $2s$ spin $\frac{1}{2}$ states, but it cannot always be looked upon as a spin $|ss\rangle$ or $|s - s\rangle$ state with respect to any choice of the Cartesian frame.

For $s > \frac{1}{2}$ an oriented state is identified as an $|sm\rangle$ state in an appropriately chosen Cartesian frame, with m taking any one of values $-s, \ldots, s$. Such a state is cylindrically symmetric with respect to the $\hat{z}$-axis which is also referred to as the axis of quantization. All other states are referred to as non-oriented states. In other words, a non-oriented state cannot be looked upon as an $|sm\rangle$ state with respect to any frame. A discussion of this idea for a system with sharp angular momentum has been done earlier [1, 13]. We now would like to characterize a coupled spin system based on these aspects. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the spin spaces of two systems with spins s_1 and s_2 respectively, then a coupled state of these two systems is a state belonging to $\mathcal{H}_1 \otimes \mathcal{H}_2$ and can be expressed as

$$|\psi_{12}\rangle = \sum_{ij} a_{ij} |\phi_i\rangle \otimes |\zeta_j\rangle \qquad \sum_{ij} |a_{ij}|^2 = 1 \tag{1}$$

where $|\phi_i\rangle$ $(i = -s_1, \ldots, s_1)$ and $|\zeta_j\rangle$ $(j = -s_2, \ldots, s_2)$ are the angular momentum basis states of the subsystems s_1 and s_2, respectively. In the study of coupling of two angular momenta, the basis states are usually chosen with respect to some common axis of quantization. For a coupled state, one possible characterization could be to relax this usual choice and ask whether a state $|\psi_{12}\rangle$ is an eigenstate for the four mutually commuting operators J_1^2, $\vec{J}_1 \cdot \hat{Q}_1$, J_2^2 and $\vec{J}_2 \cdot \hat{Q}_2$, where $\hat{Q}_1$, $\hat{Q}_2$ refer to two arbitrary axes, one for each system. While every state $|\psi_{12}\rangle$ is an eigenstate of J_1^2 and J_2^2, the eigenstates for the other two operators form a subset of the space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

It is well known that in the case of two spinors the individual eigenstates for J_i^2 and $\vec{J}_i \cdot \hat{Q}_i$ expressed in terms of basis vectors referred to the common axis of quantization $\hat{z}_0$ of a frame $x_0 y_0 z_0$ can be written in the form

$$|\psi_i\rangle = \begin{pmatrix} \cos\frac{\theta_i}{2} \\ \sin\frac{\theta_i}{2}\,\mathrm{e}^{\mathrm{i}\phi_i} \end{pmatrix}_{z_0} \qquad 0 \leqslant \theta_i \leqslant \pi \qquad 0 \leqslant \phi_i \leqslant 2\pi \qquad i = 1, 2 \tag{2}$$

where θ_i, ϕ_i are the polar angles of $\hat{Q}_i$ with respect to the common frame. The direct product state

$$\left|\psi_{12}^{(a)}\right\rangle = \begin{pmatrix} \cos\frac{\theta_1}{2} \\ \sin\frac{\theta_1}{2}\,\mathrm{e}^{\mathrm{i}\phi_1} \end{pmatrix}_{z_0} \otimes \begin{pmatrix} \cos\frac{\theta_2}{2} \\ \sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_2} \end{pmatrix}_{z_0} = \begin{pmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_1} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} \end{pmatrix}_{z_0} \tag{3}$$

is an eigenstate of J_1^2, $\vec{J}_1 \cdot \hat{Q}_1$, J_2^2 and $\vec{J}_2 \cdot \hat{Q}_2$ satisfying (with $\hbar = 1$)

$$J_1^2\left|\psi_{12}^{(a)}\right\rangle = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\left|\psi_{12}^{(a)}\right\rangle \tag{4}$$

$$\vec{J}_1 \cdot \hat{Q}_1\left|\psi_{12}^{(a)}\right\rangle = \tfrac{1}{2}\left|\psi_{12}^{(a)}\right\rangle \tag{5}$$

$$J_2^2\left|\psi_{12}^{(a)}\right\rangle = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\left|\psi_{12}^{(a)}\right\rangle \tag{6}$$

$$\vec{J}_2 \cdot \hat{Q}_2\left|\psi_{12}^{(a)}\right\rangle = \tfrac{1}{2}\left|\psi_{12}^{(a)}\right\rangle. \tag{7}$$

The other three common eigenstates are

$$\left|\psi_{12}^{(b)}\right\rangle = \begin{pmatrix} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ -\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_2} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_1} \\ -\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} \end{pmatrix}_{z_0} \tag{8}$$

$$\left|\psi_{12}^{(c)}\right\rangle = \begin{pmatrix} \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_2} \\ -\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_1} \\ -\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} \end{pmatrix}_{z_0} \tag{9}$$

$$\left|\psi_{12}^{(d)}\right\rangle = \begin{pmatrix} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ -\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_2} \\ -\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}\phi_1} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)} \end{pmatrix}_{z_0}. \tag{10}$$

From this, it is clear that every direct product state is a common eigenstate of J_1^2, $\vec{J}_1\cdot\hat{Q}_1$, J_2^2 and $\vec{J}_2\cdot\hat{Q}_2$ for some $\hat{Q}_1$ and $\hat{Q}_2$ and that given one such state, three orthogonal eigenstates can be constructed, which of course span the direct product space $\mathcal{H}_1\otimes\mathcal{H}_2$. Conversely, it follows that the common eigenstates of the above four operators have to be direct product states only.

In the case of a sharp spin s state, we have characterized a pure state as oriented if it happens to be an angular momentum state $|sm\rangle$ with respect to some quantization axis. This property can be extended to a bipartite system of two spins s_1 and s_2. A pure state $|\psi_{12}\rangle$ of a bipartite system can be regarded as oriented if

$$\vec{J}_1\cdot\hat{Q}_1\vec{J}_2\cdot\hat{Q}_2|\psi_{12}\rangle = m_1m_2|\psi_{12}\rangle \tag{11}$$

for some $\hat{Q}_1$ and $\hat{Q}_2$. If s_1 and s_2 are arbitrary, then every direct product state $|\xi_1\rangle\otimes|\xi_2\rangle$ is not necessarily oriented. However if $s_1 = s_2 = \frac{1}{2}$, every direct product state is always oriented. This follows from the homomorphism between $SU(2)$ and $O(3)$.

A direct product state of two spinors is thus non-entangled, oriented and a common eigenstate of four operators J_1^2, $\vec{J}_1\cdot\hat{Q}_1$, J_2^2, $\vec{J}_2\cdot\hat{Q}_2$. This implies that the rest of the states in the Hilbert space $\mathcal{H}_1\otimes\mathcal{H}_2$ do not share the above properties. They are not only non-oriented but also possess entanglement. Let us now look at the nature of spin–spin correlations present in these coupled states.

The spin–spin correlations according to Kitagawa and Ueda are responsible for the existence of squeezing. In their paper [9], they suggest that every state of a spin s system can be visualized as a coupled state of $2s$ spinor states and claim that the squeezing behaviour of the spin s state is due to the correlations that exist among the $2s$ spinor states. In our earlier paper [1, 2], we have indeed given an explicit method of construction of a general pure spin s state in terms of $2s$ spinor states. Based on this construction, we have analysed the nature of a squeezed spin s state and in the case of $s = 1$, we have shown that the squeezing aspect is intimately connected with the pairwise correlations defined through

$$C_{\mu\nu}^{ij}(s) = \left\langle S_\mu^i S_\nu^j\right\rangle - \left\langle S_\mu^i\right\rangle\left\langle S_\nu^j\right\rangle \qquad i, j = 1, 2, \ldots, 2s \tag{12}$$

where S_μ^i is the μth component of the ith spin $\vec{S}^i$. We wish to call these self-correlations. While these are absent in the case of a single spinor, there would be a large number of such correlations for large s. These coupled states of arbitrary spin s_1 and s_2 not only possess the above 'self-correlations' $C_{\mu\nu}^{ij}(s_1)$ and $C_{\mu\nu}^{ij}(s_2)$ but also the 'mutual correlations' between s_1 and s_2. These mutual correlations can be defined in an analogous way as

$$D_{\mu\nu}^{12} = \langle S_{1\mu}S_{2\nu}\rangle - \langle S_{1\mu}\rangle\langle S_{2\nu}\rangle \tag{13}$$

where $S_{1\mu}(S_{2\nu})$ is the μth (νth) component of the spin vector $\vec{S}_1(\vec{S}_2)$. For a direct product state of two spinors, it is easy to see that both $C_{\mu\nu}^{ij}$ and $D_{\mu\nu}^{12}$ are zero. A direct product state with s_1 or s_2 exceeding $\frac{1}{2}$ may possess self-correlations but there are no mutual correlations. An entangled pure state, on the other hand, certainly always possesses mutual correlations, although there may or may not be self-correlations. Our work in this paper is limited to the discussion of a coupled state of two spinors and we will be using the ideas presented here to arrive at the right criterion for the squeezing of such a bipartite state.

It may be mentioned here that the most general state of two spins s_1 and s_2 is a mixed state which is not only entangled but also rich in both self and mutual correlations. In addition, it may possess statistical correlations owing to the distribution of the spins in it.

3. Squeezing criterion for a general bipartite state

The problem of identifying squeezing in quantum systems other than the radiation field has been taken up in the past and it is interesting to note that the characterization of squeezing in such systems has been done using different approaches. For example, considering two two-level atomic states, Barnett and Dupertuis [5] have constructed squeezed atomic states in analogy with the multimode squeezed states and thermofield states of the radiation field. They have identified correlated states, which are actually entangled, and define the collective atomic dipole operators for them and these operators obey the usual angular momentum algebra. The basis for identifying squeezing of atomic states is then the inequality relationships resulting from the Heisenberg uncertainty relationship satisfied by the dipole operators. Wineland *et al* [7] in their detailed study examine again correlated or entangled states of atomic systems and characterize squeezing through the reduction of projection noise in the context of Ramsey spectroscopy. As far as spin systems are concerned, Kitagawa and Ueda [9] have also attempted to associate squeezing with quantum correlations after critically examining other criteria that have been given in the literature.

According to Kitagawa and Ueda, a spin state $|\phi\rangle$ is said to be squeezed if in that state

$$\Delta(\vec{S}\cdot\hat{n}_\perp)^2 < \frac{|\langle\vec{S}\cdot\hat{n}\rangle|}{2} \tag{14}$$

where $\hat{n}$ is a unit vector along $\langle\vec{S}\rangle$, called the mean spin direction and $\hat{n}_\perp$ is orthogonal to $\hat{n}$. This condition clearly distinguishes a squeezed state from other states in an intrinsic way.

The task now is to extend this to the case of a bipartite system which in general does not possess a sharp value of the total spin. For example, a coupled state of two spinors can be a superposition of triplet ($s = 1$) as well as the singlet states ($s = 0$). A possible choice for the criterion is to consider the spin components $S_{1\mu} + S_{2\mu}$ of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ with respect to a common frame $x_0y_0z_0$ and define the criterion exactly as in equation (14) with the understanding that $\hat{n}$ denotes the direction of $\langle\vec{S}_1 + \vec{S}_2\rangle$ and $\vec{S}.\hat{n}$, $\vec{S}.\hat{n}_\perp$ denote the components of $\vec{S} = \vec{S}_1 + \vec{S}_2$ along and perpendicular to $\hat{n}$, respectively. We have looked into this choice which leads to a conclusion that certain direct product states such as

$$|\psi\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} \\ -\frac{1}{4} \end{pmatrix} \tag{15}$$

will possess squeezing if we consider such a choice. This is undesirable since the two subsystems may be totally independent, non-squeezed and non-interacting and if we formally define direct product states of the two, these would be squeezed if we employ the criterion suggested above.

We would therefore like to search for an appropriate criterion for squeezing of bipartite states taking the above aspect into consideration. As an aid in this direction, it may be noted here that in problems aimed at determining correlations in a bipartite system, when the subsystems are physically separated, two observers make measurements in frames of their own choice. This freedom of choice of frame, in the context of entanglement, arises from the property that entanglement of a bipartite system is invariant under local rotation of frames (local unitary transformations on individual states) describing the subsystems. It is therefore necessary to allow for this freedom of choice of local frames for discussing squeezing and this is done as follows.

Suppose a bipartite state $|\psi_{12}\rangle$ (equation (1)) of two spins is specified with respect to some frame $x_0y_0z_0$ (say). Suppose $[\hat{n}_1, \hat{n}_{1\perp}, \hat{n}_{1\perp'} = \hat{n}_1 \times \hat{n}_{1\perp}]$ and $[\hat{n}_2, \hat{n}_{2\perp}, \hat{n}_{2\perp'} = \hat{n}_2 \times \hat{n}_{2\perp}]$ denote two sets of mutually orthogonal directions. Now it is easy to see that the components of spin operators $\vec{S}_1$ and $\vec{S}_2$ with respect to these triplets satisfy the usual angular momentum commutation relations,

$$[\vec{S}_1 \cdot \hat{n}_{1\perp} + \vec{S}_2 \cdot \hat{n}_{2\perp}, \vec{S}_1 \cdot \hat{n}_{1\perp'} + \vec{S}_2 \cdot \hat{n}_{2\perp'}] = \mathrm{i}(\vec{S}_1 \cdot \hat{n}_1 + \vec{S}_2 \cdot \hat{n}_2) \tag{16}$$

and the uncertainty relationship for these operators takes the form

$$\Delta(\vec{S}_1 \cdot \hat{n}_{1\perp} + \vec{S}_2 \cdot \hat{n}_{2\perp})^2 \Delta(\vec{S}_1 \cdot \hat{n}_{1\perp'} + \vec{S}_2 \cdot \hat{n}_{2\perp'})^2 \geqslant \frac{(\langle \vec{S}_1 \cdot \hat{n}_1 \rangle + \langle \vec{S}_2 \cdot \hat{n}_2 \rangle)^2}{4} \tag{17}$$

where,

$$\Delta(\vec{S}_1 \cdot \hat{a} + \vec{S}_2 \cdot \hat{b})^2 = \langle \psi_{12} | (\vec{S}_1 \cdot \hat{a} + \vec{S}_2 \cdot \hat{b})^2 | \psi_{12} \rangle - \langle \psi_{12} | (\vec{S}_1 \cdot \hat{a} + \vec{S}_2 \cdot \hat{b}) | \psi_{12} \rangle^2. \tag{18}$$

Suppose now $\hat{n}_1$ and $\hat{n}_2$ denote the mean spin directions of the individual spinor states, defined through

$$\hat{n}_i = \frac{\langle \psi_{12} | \vec{S}_i | \psi_{12} \rangle}{|\langle \psi_{12} | \vec{S}_i | \psi_{12} \rangle|} \qquad i = 1, 2 \tag{19}$$

and $\hat{n}_{i\perp}$ are directions such that

$$\hat{n}_{i\perp} \cdot \hat{n}_i = 0 \qquad i = 1, 2. \tag{20}$$

These directions $(\hat{n}_i, \hat{n}_{i\perp}, \hat{n}_{i\perp'})$ define the individual Lakin frames [1] as we have

$$\langle \vec{S}_i \cdot \hat{n}_{i\perp} \rangle = 0 \qquad \langle \vec{S}_i \cdot \hat{n}_{i\perp'} \rangle = 0 \qquad i = 1, 2. \tag{21}$$

The criterion we adopt is as follows. Given a bipartite state, there are two mean spin directions $\hat{n}_1$ and $\hat{n}_2$ defined through equation (19). A bipartite state with mean spin directions $\hat{n}_1, \hat{n}_2$ is said to be squeezed in a perpendicular component $\vec{S}_1 \cdot \hat{n}_{1\perp} + \vec{S}_2 \cdot \hat{n}_{2\perp}$, if the variance of this operator in the given state is less than half the sum of the absolute values of the expectation values of the spin vectors in that state.

Expressed mathematically, the criterion becomes

$$\Delta\left(\vec{S}_1 \cdot \hat{n}_{1_\perp} + \vec{S}_2 \cdot \hat{n}_{2_\perp}\right)^2 < \frac{|\langle \vec{S}_1 \rangle| + |\langle \vec{S}_2 \rangle|}{2}. \tag{22}$$

This can be further written in the form

$$\Delta\left(\vec{S}_1 \cdot \hat{n}_{1_\perp}\right)^2 + \Delta\left(\vec{S}_2 \cdot \hat{n}_{2_\perp}\right)^2 + 2\left\langle \vec{S}_1 \cdot \hat{n}_{1_\perp} \otimes \vec{S}_2 \cdot \hat{n}_{2_\perp} \right\rangle < \frac{|\langle \vec{S}_1 \cdot \hat{n}_1 \rangle| + |\langle \vec{S}_2 \cdot \hat{n}_2 \rangle|}{2}. \tag{23}$$

Before we provide some justification for this criterion, it must be noted that it is in an invariant form so that given any frame, the criterion can be expressed in terms of the spin components referred to that frame, once the directions $\hat{n}_i, \hat{n}_{i\perp}$ are determined in that frame. Instead, one can also transform the frame by appropriate rotations and obtain the individual Lakin frames.

Suppose the so obtained frames $x_1y_1z_1$ and $x_2y_2z_2$ are named such that $\hat{z}_i = \hat{n}_i$, $\hat{x}_i = \hat{n}_{i\perp}$ and $\hat{y}_i = \hat{n}_{i\perp'}$, the criterion takes the forms

$$\Delta\left(S_{1x_1}\right)^2 + \Delta\left(S_{2x_2}\right)^2 + 2\left\langle S_{1x_1} \otimes S_{2x_2}\right\rangle < \frac{\left|\left\langle \vec{S}_{1z_1}\right\rangle\right| + \left|\left\langle \vec{S}_{2z_2}\right\rangle\right|}{2} \tag{24}$$

$$\Delta\left(S_{1y_1}\right)^2 + \Delta\left(S_{2y_2}\right)^2 + 2\left\langle S_{1y_1} \otimes S_{2y_2}\right\rangle < \frac{\left|\left\langle \vec{S}_{1z_1}\right\rangle\right| + \left|\left\langle \vec{S}_{2z_2}\right\rangle\right|}{2}. \tag{25}$$

The given state would therefore be squeezed in the components $S_{1x_1} + S_{2x_2}$ or $S_{1y_1} + S_{2y_2}$ if equation (24) or (25) is satisfied. For a bipartite system of two spinors, the criterion reduces to a simpler form since,

$$\Delta S_{1x_1}^2 = \Delta S_{2x_2}^2 = \Delta S_{1y_1}^2 = \Delta S_{2y_2}^2 = \tfrac{1}{4} \tag{26}$$

always and therefore the squeezing criterion along the individual x- and y-axes reduces to

$$\left\langle S_{1x_1} \otimes S_{2x_2}\right\rangle < \frac{\left|\left\langle \vec{S}_{1z_1}\right\rangle\right| + \left|\left\langle \vec{S}_{2z_2}\right\rangle\right| - 1}{4} \tag{27}$$

$$\left\langle S_{1y_1} \otimes S_{2y_2}\right\rangle < \frac{\left|\left\langle \vec{S}_{1z_1}\right\rangle\right| + \left|\left\langle \vec{S}_{2z_2}\right\rangle\right| - 1}{4}. \tag{28}$$

We wish to now apply this criterion to bipartite states of interest and see whether they are squeezed or not. To begin with let us consider a bipartite state which is a direct product state of two states with the first being $\left|\frac{1}{2}\frac{1}{2}\right\rangle$ with respect to $\hat{z}_1$ while the second is $\left|\frac{1}{2}\frac{1}{2}\right\rangle$ with respect to $\hat{z}_2$. When expressed in terms of the special common frame $x_0y_0z_0$ [1] and in individual Lakin frames, it has the structure

$$|\psi_{12}\rangle = |\xi_1\rangle \otimes |\xi_2\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ -\sin^2\frac{\theta}{2} \end{pmatrix}_{z_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{z_1} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{z_2}. \tag{29}$$

Since

$$\Delta\left(\vec{S}_1 \cdot \hat{n}_{1_\perp}\right)^2 = \Delta\left(\vec{S}_2 \cdot \hat{n}_{2_\perp}\right)^2 = \tfrac{1}{4} \tag{30}$$

$$\left\langle \vec{S}_1 \cdot \hat{n}_{1_\perp} \otimes \vec{S}_2 \cdot \hat{n}_{2_\perp}\right\rangle = \left\langle \vec{S}_1 \cdot \hat{n}_{1_\perp}\right\rangle \otimes \left\langle \vec{S}_2 \cdot \hat{n}_{2_\perp}\right\rangle = 0 \tag{31}$$

$$|\langle \vec{S}_1 \cdot \hat{z}_1\rangle| = |\langle \vec{S}_2 \cdot \hat{z}_2\rangle| = \tfrac{1}{2} \tag{32}$$

the criterion in the form (23) or in the forms (24), (25) is not satisfied at all and hence a direct product state of two spinors is never squeezed. This is in perfect agreement with Kitagawa and Ueda [9] in that the squeezing arises due to correlations and a direct product state which has neither self nor mutual correlations is therefore not squeezed.

3.1. Squeezing of entangled pure state

It is clear from the previous discussion that entanglement is necessary in the case of a two-spinor bipartite state for squeezing to occur. However, it is to be seen whether entanglement is sufficient also. We therefore begin this study by considering a general pure state which is entangled. Such a state can be expressed with respect to a basis $|m_1 m_2\rangle_{z_0}$ referred to a common frame $x_0 y_0 z_0$ in the form

$$|\psi_{12}\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} \qquad \sum_{ij} |a_{ij}|^2 = 1 \tag{33}$$

where of course $a_{11}a_{22} \neq a_{12}a_{21}$. This latter condition ensures [11] that $|\psi_{12}\rangle$ is entangled. Since, in the general case, the frame $x_0 y_0 z_0$ may not be a Lakin frame for either spinor, we consider the rotation

$$R_{12} = R_1(\phi_1, \theta_1, 0) \otimes R_2(\phi_2, \theta_2, 0) \tag{34}$$

on this state so that

$$|\psi_{12}\rangle \longrightarrow |\psi'_{12}\rangle = R_{12}|\psi_{12}\rangle = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} \tag{35}$$

where c_{11} can be taken to be a non-negative real number (by using the freedom of choice of the overall phase). The individual rotations on the coordinate system $x_0 y_0 z_0$ take it to the respective Lakin frames $x_1 y_1 z_1$ and $x_2 y_2 z_2$ if the Euler angles of rotation are chosen to satisfy

$$\tan\phi_i = \frac{\langle S_{iy}\rangle}{\langle S_{ix}\rangle} \qquad i = 1, 2 \tag{36}$$

$$\tan\theta_i = \frac{(\langle S_{iy}\rangle^2 + \langle S_{ix}\rangle^2)^{\frac{1}{2}}}{\langle S_{iz}\rangle} \qquad i = 1, 2. \tag{37}$$

With these transformations, we now obtain

$$\langle S_{1x_1}\rangle = \langle R_{12} S_{1x} R_{12}^\dagger\rangle = \tfrac{1}{2}(c_{11}c_{21} + c_{12}^\star c_{22} + c_{22}^\star c_{12} + c_{21}^\star c_{11}) = 0 \tag{38}$$

$$\langle S_{2x_2}\rangle = \langle R_{21} S_{2x} R_{21}^\dagger\rangle = \tfrac{1}{2}(c_{11}c_{12} + c_{12}^\star c_{11} + c_{21}^\star c_{22} + c_{22}^\star c_{21}) = 0 \tag{39}$$

$$\langle S_{1y_1}\rangle = \langle R_{12} S_{1y} R_{12}^\dagger\rangle = \frac{\mathrm{i}}{2}(-c_{11}c_{21} - c_{12}^\star c_{22} + c_{21}^\star c_{11} + c_{22}^\star c_{12}) = 0 \tag{40}$$

$$\langle S_{2y_2}\rangle = \langle R_{21} S_{2y} R_{21}^\dagger\rangle = \frac{\mathrm{i}}{2}(-c_{11}c_{12} + c_{12}^\star c_{11} - c_{21}^\star c_{22} + c_{22}^\star c_{21}) = 0 \tag{41}$$

$$\langle S_{1z_1}\rangle = \langle R_{12} S_{1z} R_{12}^\dagger\rangle = \tfrac{1}{2}\big(c_{11}^2 + |c_{12}|^2 - |c_{21}|^2 - |c_{22}|^2\big) \tag{42}$$

$$\langle S_{2z_2}\rangle = \langle R_{21} S_{2z} R_{21}^\dagger\rangle = \tfrac{1}{2}\big(c_{11}^2 - |c_{12}|^2 + |c_{21}|^2 - |c_{22}|^2\big). \tag{43}$$

A glance at the squeezing criterion implies that for the state to exhibit squeezing, first of all, $\langle S_{1z_1}\rangle, \langle S_{2z_2}\rangle \neq 0$. Further, the first four equations referring to the Lakin frame yield

$$c_{11}c_{21} = -c_{12}^\star c_{22} \tag{44}$$

$$c_{11}c_{12} = -c_{21}^\star c_{22} \tag{45}$$

which lead to

$$c_{11}c_{22}\big(|c_{21}|^2 - |c_{12}|^2\big) = 0 \tag{46}$$

and

$$c_{12}^{\star}c_{21}\left(c_{11}^{2}-|c_{22}|^{2}\right)=0. \tag{47}$$

There arise several cases satisfying these conditions

$$\text{case 1:}\quad c_{ij}=\delta_{ii_0}\delta_{jj_0}\qquad \text{for fixed}\quad i_0,\, j_0 \tag{48}$$

$$\text{case 2:}\quad |c_{21}|=|c_{12}|\neq 0\qquad |c_{11}|=|c_{22}|\qquad \phi_{12}=\pi+\phi_{22}-\phi_{21} \tag{49}$$

$$\text{case 3:}\quad c_{11}=c_{22}=0\qquad c_{12},c_{21}\neq 0 \tag{50}$$

$$\text{case 4:}\quad |c_{12}|=|c_{21}|=0. \tag{51}$$

The first case refers to direct product states. The second case implies $\langle S_{1z_1}\rangle=\langle S_{2z_2}\rangle=0$ and hence although it is entangled, it is not squeezed. This state is actually a singlet state with total spin $s=0$ and $|\langle S_{1z_1}\rangle|+|\langle S_{2z_2}\rangle|=0$.

It is this latter value which makes a singlet state not squeezed although there is entanglement. We therefore conclude that entanglement is necessary but not sufficient for squeezing to be present. Consider now the state in case (3), which has the form

$$|\xi\rangle=\begin{pmatrix}0\\ c_{12}\\ c_{21}\\ 0\end{pmatrix}. \tag{52}$$

Under the rotation $R_1\left(0,\frac{\pi}{2},0\right)\otimes I$, which is a local rotation, this state changes to

$$|\xi'\rangle=(R_1\otimes I)|\xi\rangle=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0\\ c_{12}\\ c_{21}\\ 0\end{pmatrix}=\begin{pmatrix}c_{21}\\ 0\\ 0\\ -c_{12}\end{pmatrix} \tag{53}$$

which shows that it belongs to case (4). It is therefore enough to consider only the states belonging to case (4). Here too, if we use the degree of freedom for the overall phase, the normalization condition and the freedom of rotation about the respective z_i axes, the state can be reduced to the simple form with its elements parametrized as

$$|\chi\rangle=\begin{pmatrix}\cos\frac{\theta}{2}\\ 0\\ 0\\ \sin\frac{\theta}{2}\end{pmatrix}\qquad 0<\theta<\pi. \tag{54}$$

This is the simplest matrix form of an entangled state with non-zero mean values for its individual spin vectors. The relevant quantities needed to determine the squeezing behaviour are

$$\Delta S_{1x_1}^2=\Delta S_{2x_2}^2=\Delta S_{1y_1}^2=\Delta S_{2y_2}^2=\tfrac{1}{4}$$

$$\langle S_{1x_1}\otimes S_{2x_2}\rangle=\tfrac{1}{4}\sin\theta=-\langle S_{1y_1}\otimes S_{2y_2}\rangle$$

$$|\langle S_{1_{z_1}}\rangle|=|\langle S_{2_{z_2}}\rangle|=\tfrac{1}{2}|\cos\theta|.$$

Substituting these in the squeezing condition given by equation (23) or (24), we obtain

$$1+\sin\theta<|\cos\theta|\qquad \text{for}\quad S_{1x_1}+S_{2x_2} \tag{55}$$

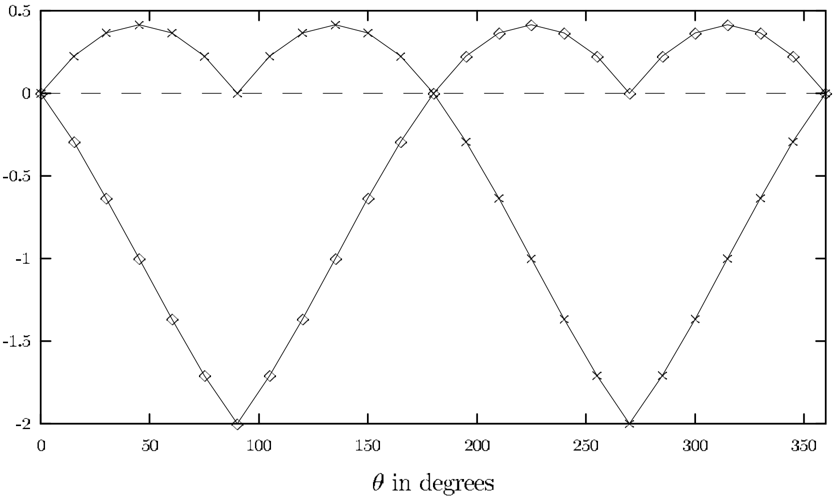

Figure 1. Variation of squeezing Q_x ($\Diamond$) and Q_y ($\times$) with respect to θ.

and

$$1 - \sin\theta < |\cos\theta| \qquad \text{for} \quad S_{1y_1} + S_{2y_2}. \tag{56}$$

These conditions are certainly satisfied for a wide range of θ which indicates that a wide variety of states of the form of equation (54) with different values of θ exhibit squeezing. The variation of squeezing with respect to θ is shown in figure 1 where we have plotted the difference between the right- and left-hand sides of equations (55) and (56)

$$Q_x = |\cos\theta| - \sin\theta - 1 \tag{57}$$

$$Q_y = |\cos\theta| + \sin\theta - 1 \tag{58}$$

as a function of θ. Positive (negative) values of Q_x, Q_y show the presence (absence) of squeezing. It may be noted from figure 1 that $\theta = 90^\circ$ corresponds to the singlet state referred to in case (2).

At this stage, it is relevant to see how the correlations account for squeezing. As mentioned earlier, a bipartite system of two spinors has no self-correlations. The mutual correlations which exist between the two spins are

$$D^{12}_{\mu_1\nu_2} = \langle S_{1\mu_1} S_{2\nu_2}\rangle - \langle S_{1\mu_1}\rangle\langle S_{2\nu_2}\rangle \qquad \mu, \nu = x, y, z. \tag{59}$$

If these are zero, then the state has no mutual spin–spin correlation. All direct product states fall into this category. For the squeezed states defined by equation (54), these correlations with respect to the Lakin frames turn out to be

$$D^{12}_{x_1x_2} = -D^{12}_{y_1y_2} = \tfrac{1}{4}\sin\theta \tag{60}$$

$$D^{12}_{z_1z_2} = \tfrac{1}{4}\sin^2\theta \tag{61}$$

$$D^{12}_{x_1y_2} = D^{12}_{x_1z_2} = D^{12}_{y_1z_2} = 0. \tag{62}$$

The graphs of figure 2, in which we have plotted both the squeezing and mutual correlation functions, reveal that squeezing is absent whenever there are no mutual correlations and when the mutual correlations assume their extreme values. Indeed these extreme values correspond

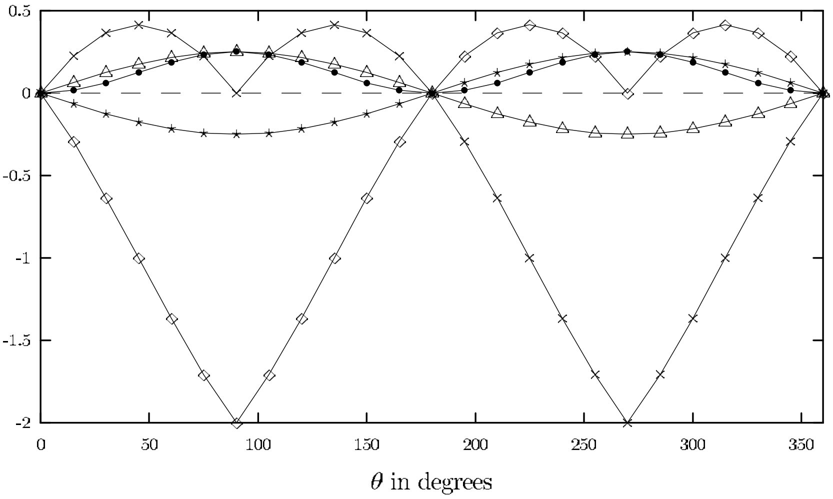

Figure 2. Variation of spin–spin correlations D_{xx}^{12} (△), D_{yy}^{12} (★) and D_{zz}^{12} (●), The plot also shows squeezing in Q_x (◇) and Q_y (×) with respect to θ.

to a singlet state which confirms that it is a maximally entangled state. However, it has both $\langle\vec{S}_1\rangle = \langle\vec{S}_2\rangle = 0$ due to which it does not exhibit squeezing.

4. Generation of squeezing

We have so far discussed the nature of spin squeezing for a general coupled pure state in terms of a new criterion which is in some sense a generalization of the earlier criterion. It is of interest to know whether squeezing can be generated by subjecting a spin system to external interactions. Indeed, such attempts have been done in the literature for atomic systems [5] and for spin systems [7, 9]. For sharp spin states, Kitagawa and Ueda consider a Hamiltonian quadratic in the spin operators and show that the state generated during evolution under such a Hamiltonian possesses squeezing. It is clear that a Hamiltonian linear in the spin variables can only lead to a rotation of the state or of the coordinate system. Such an action does not introduce any squeezing. It is therefore necessary that the Hamiltonian should be at least quadratic in the spin variables in the case of single spin. For a bipartite system on the other hand, it should be at least a linear combination of the product of the individual spin operators. We therefore consider below a spin–spin interaction Hamiltonian,

$$H = \mathrm{i}\eta[S_{1+}S_{2+} - S_{1-}S_{2-}] \tag{63}$$

where

$$S_{i\pm} = S_{ix} \pm \mathrm{i}S_{iy} \qquad i = 1, 2 \tag{64}$$

and η is any real number. This Hamiltonian incidentally is a special case of the Hamiltonian in [5] if one makes the identification suggested by Feynman *et al* [14] that any two-level quantum system is equivalent to a spin $\frac{1}{2}$ system. If the initial coupled state $|\psi\rangle$ of these two spinors is chosen to be a direct product state $|\psi\rangle$ as in equation (29), where the basis vectors and the Hamiltonian are referred to the frame $x_0y_0z_0$ [1], the evolution in time given by

$$|\psi(t)\rangle = \exp(-\mathrm{i}Ht)|\psi\rangle = \exp[\eta t(S_{1+}S_{2+} - S_{1-}S_{2-})]|\psi\rangle \tag{65}$$

leads to the explicit form

$$|\psi(t)\rangle = \begin{pmatrix} \cos\tau\cos^2\frac{\theta}{2} - \sin\tau\sin^2\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ -\sin\tau\cos^2\frac{\theta}{2} - \cos\tau\sin^2\frac{\theta}{2} \end{pmatrix} \qquad \tau = \eta t. \tag{66}$$

This state is in general entangled. As in any general case, we find here also that the frame $x_0y_0z_0$ is not the common Lakin frame for the two spinors, since

$$\langle S_{1x_0}\rangle = \frac{\sin\theta}{2}[\cos\tau + \sin\tau\cos\theta] = -\langle S_{2x_0}\rangle \tag{67}$$

$$\langle S_{1y_0}\rangle = \langle S_{2y_0}\rangle = 0 \tag{68}$$

$$\langle S_{1z_0}\rangle = \langle S_{2z_0}\rangle = \frac{1}{2}\left[\cos 2\tau\cos\theta - \frac{1}{2}\sin 2\tau\sin^2\theta\right]. \tag{69}$$

It is easier to analyse the squeezing behaviour if we go over to the individual Lakin frames $x_1y_1z_1$ and $x_2y_2z_2$ via the rotations through

$$\alpha_i = \tan^{-1}\left(\frac{\langle S_{ix_0}\rangle}{\langle S_{iz_0}\rangle}\right) \qquad i = 1, 2 \tag{70}$$

of $x_0y_0z_0$ about the y_0-axis.

In these Lakin frames, the expectation values of the various spin operators are given by

$$\langle S_{ix_i}\rangle = \langle S_{ix_0}\rangle\cos\alpha_i - \langle S_{iz_0}\rangle\sin\alpha_i = 0 \tag{71}$$

$$\langle S_{iy_i}\rangle = \langle S_{iy_0}\rangle = 0 \tag{72}$$

$$\langle S_{iz_i}\rangle = \langle S_{ix_0}\rangle\sin\alpha_i + \langle S_{iz_0}\rangle\cos\alpha_i \tag{73}$$

$$\Delta S_{1x_1}^2 + \Delta S_{2x_2}^2 + 2\langle S_{1x_1}\otimes S_{2x_2}\rangle = \tfrac{1}{2}(1 - A\cos^2\alpha_1 - B\sin 2\alpha_1\sin\theta - \cos^2\theta\sin^2\alpha_1) \tag{74}$$

$$\Delta S_{1y_1}^2 + \Delta S_{2y_2}^2 + 2\langle S_{1y_1}\otimes S_{2y_2}\rangle = \tfrac{1}{2} + \tfrac{1}{2}[\sin 2\tau\cos\theta - \sin^2\theta\sin^2\tau] \tag{75}$$

$$|\langle S_{1z_1}\rangle| + |\langle S_{2z_2}\rangle| = \left(\sin^2\theta(\cos\tau + \sin\tau\cos\theta)^2 + \left(\cos\theta\cos 2\tau - \tfrac{1}{2}\sin 2\tau\sin^2\theta\right)^2\right)^{\frac{1}{2}} \tag{76}$$

where $A = (\cos\theta\sin 2\tau + \sin^2\theta\cos^2\tau)$ and $B = \sin\tau - \cos\theta\cos\tau$. The quantities Q_x, Q_y defined earlier now become functions of time and are given by

$$Q_x(t) = \left(\sin^2\theta(\cos\tau + \sin\tau\cos\theta)^2 + \left(\cos\theta\cos 2\tau - \tfrac{1}{2}\sin 2\tau\sin^2\theta\right)^2\right)^{\frac{1}{2}} \\ - (1 - A\cos^2\alpha_1 - B\sin 2\alpha_1\sin\theta - \cos^2\theta\sin^2\alpha_1) \tag{77}$$

$$Q_y(t) = \left(\sin^2\theta(\cos\tau + \sin\tau\cos\theta)^2 + \left(\cos\theta\cos 2\tau - \tfrac{1}{2}\sin 2\tau\sin^2\theta\right)^2\right)^{\frac{1}{2}} \\ - (1 + \sin 2\tau\cos\theta - \sin^2\theta\sin^2\tau). \tag{78}$$

We infer that the state is squeezed if either $Q_x(t)$ or $Q_y(t)$ is positive. The graphs of $Q_x(t)$ and $Q_y(t)$ plotted below in figure 3 show that the squeezing is observed for a wide range of values of θ and τ, except at certain points. These points are at $\tau = 90°$, $\theta = \frac{n\pi}{2}$, $n = 0, 1, 2, \ldots$ and for $\tau = 45°$, $\theta = 0$, etc, and correspond to either direct product or singlet states.

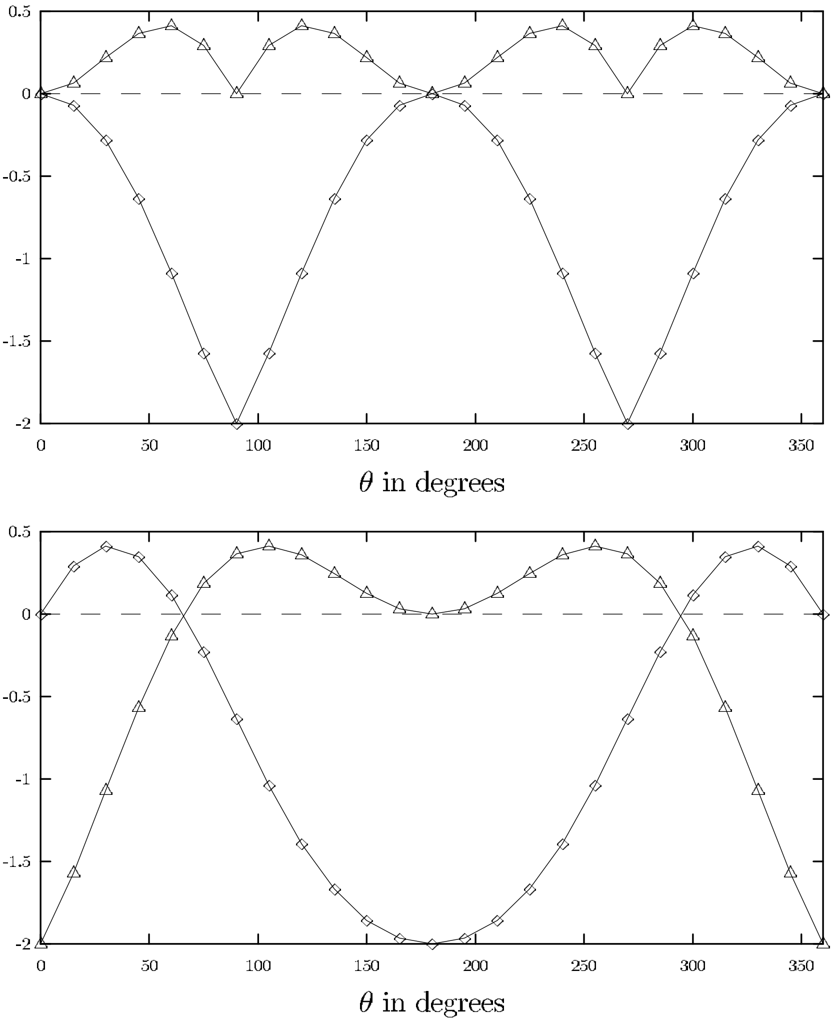

Figure 3. Variation of squeezing Q_x ($\Diamond$) and Q_y ($\triangle$) with respect to θ with $\tau = 90°$ in the upper plot and $\tau = 45°$ in the lower plot.

We now look at the mutual correlations that exist between the two spinors at various instants of time evolution. These are explicitly given by

$$D^{12}_{x_1x_2} = -\tfrac{1}{4}(A\cos^2\alpha_1 + B\sin 2\alpha_1 \sin\theta + \cos^2\theta\sin^2\alpha_1) \tag{79}$$

$$D^{12}_{y_1y_2} = \tfrac{1}{4}(\sin 2\tau\cos\theta - \sin^2\theta\sin^2\tau) \tag{80}$$

$$D^{12}_{z_1z_2} = \tfrac{1}{4}\big[A\sin^2\alpha_1 - B\sin\theta\sin 2\alpha_1 + \cos^2\theta\cos^2\alpha_1 - \big(\sin^2\theta(\cos\tau + \sin\tau\cos\theta)^2 + 4\big(\cos\theta\cos 2\tau - \tfrac{1}{2}\sin 2\tau\sin^2\theta\big)^2\big)\big] \tag{81}$$

$$D^{12}_{x_1z_2} = -\frac{\sin 2\alpha_1}{8}(B^2 - \sin^2\theta) - \frac{B\sin\theta\cos 2\alpha_1}{4} \tag{82}$$

$$D^{12}_{x_1y_2} = D^{12}_{y_1z_2} = 0. \tag{83}$$

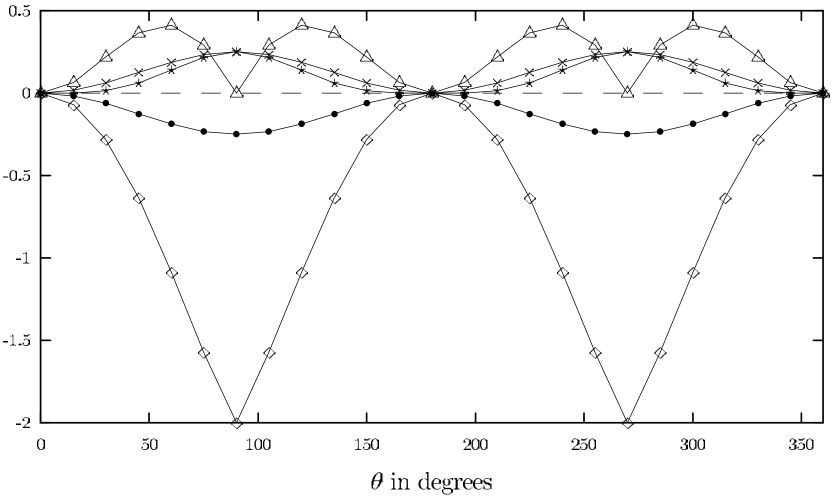

Figure 4. Variation of spin–spin correlations $D^{12}_{x_1x_2}$ ($\times$), $D^{12}_{y_1y_2}$ ($\bullet$) and $D^{12}_{z_1z_2}$ ($\bigstar$). The plot also shows squeezing in Q_x ($\Diamond$) and Q_y ($\triangle$) with respect to θ with $\tau = 90°$.

Plotting these correlations together with squeezing functions, we observe from figure 4 below that whenever there is squeezing the state necessarily possesses spin–spin correlations. The graphs lead to similar conclusions to those arrived at earlier in the discussion of the general case.

5. Summary

We have looked into the squeezing aspect of a pure bipartite state consisting of two spinors. A suitable criterion for the squeezing of such states has been obtained which is a generalization of the squeezing criterion for states of sharp spin. While squeezing is established in the case of sharp spins due to self-correlations, that for a bipartite state occurs due to the presence of both self and mutual correlations. The existence of mutual correlations also implies entanglement. This raises the question whether every entangled state is squeezed. We have shown that all entangled states of two spinors are squeezed except the singlet state which is an exception. This state lacks squeezing since it has both $\langle \vec{S}_1 \rangle = 0$, $\langle \vec{S}_2 \rangle = 0$. A direct product of two spinors, on the other hand, has neither self nor mutual correlations and hence is never squeezed. However, if s_1 or $s_2 > \frac{1}{2}$, then a direct product state can indeed possess self-correlations and such a bipartite state may show squeezing. An example of this could be the direct product of a spin 1 squeezed state with a spin $\frac{1}{2}$ state. These situations indicate that while entanglement stems from only mutual correlations, squeezing arises due to both of them and when there is net mean spin value for either of the subsystems.

Our study in this paper gives some justification to some of the claims made by Kitagawa and Ueda [9] regarding what exactly causes squeezing and how squeezed states can be generated. We have shown that spin–spin interactions can lead to entangled as well as squeezed states. Further studies on these aspects are under progress where we are also planning to analyse the squeezing of mixed states of bipartite systems in which there are not only quantum correlations among and within the spins but also correlations arising due to the nature of statistical distribution in the individual spin assemblies.

It may also be noted here that Trifonov in his paper [15] uses the Schrödinger–Robertson uncertainty relationship instead of the Heisenberg uncertainty relationship as the basis to define what are called generalized intelligent states [GIS], which according to him exhibit squeezing. In this context, it is interesting to see how squeezing for bipartite systems can be pictured based on the above more general inequality relationship. We hope to address this and related issues in our future work.

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